

DOI:10.19431/j.cnki.1673-0062.2019.02.011

胀缩渗透圆形管道内多解的构造

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摘要:多解存在于胀缩渗透圆形管道内的流体流动问题中。基于奇异摄动方法,给出了关于多解的渐近解。数值解与渐近解进行了比较,结果表明数值解与渐近解吻合的很好,说明所构造的渐近解是可靠且有效的。这样不仅可以利用此渐近解去拓展基于血液流的胀缩渗透圆形管道内的研究,而且也丰富了对多解的认识,有助于掌握血液在血管内的流动规律,对治疗心脑血管等病具有重要的借鉴意义。

关键词:胀缩渗透;多解;渐近解

中图分类号: O242

文献标志码: A

文章编号: 1673-0062(2019)02-0070-05

The Construction of Multiple Solutions Arising from a Porous Pipe with Expanding or Contracting Wall

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Abstract: Multiple solutions exist in laminar flow arising from a porous pipe with expanding or contracting wall. Based on a singular perturbation method, asymptotic solutions are obtained. Asymptotic solutions agree well with numerical solutions, indicating that the asymptotic solutions are efficient. It not only provides how to construct multiple solutions, but also illustrates the corresponding mechanism, which is helpful to understand the current problems.

key words: expanding or contracting and porous; multiple solutions; asymptotic solutions

0 引言

考虑一半无限长渗透胀缩圆形管道^[1-3]内的非稳态不可压缩的牛顿流体流动^[4-5]。管道半径

为 $a(t)$,壁面具有均匀的渗透速度 v_w ,壁面膨胀或收缩的速率为 $\dot{a}(t)$ 。如图1所示,以管道中心为坐标原点建立坐标系,坐标轴 z, r 分别平行和垂直于管道壁面, u, v 分别为 z, r 方向上的速度

收稿日期:2018-11-11

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分量。

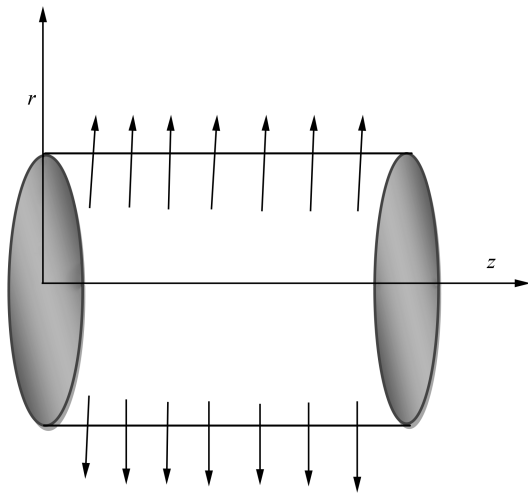


图1 胀缩渗透圆形管道流动模型

Fig.1 Flow model of dilatation-shrinkage permeable circular pipeline

描述这一物理模型在二维柱坐标系下的 Navier-Stokes 方程为:

$$u_z + v_r + \frac{v}{r} = 0 \quad (1)$$

$$u_t + uu_z + vv_r = -\frac{p_z}{\rho} + \nu(u_{zz} + u_{rr} + \frac{u_r}{r}) \quad (2)$$

$$v_t + uv_z + vv_r = -\frac{p_r}{\rho} + \nu(v_{zz} + v_{rr} + \frac{v_r}{r} - \frac{v}{r^2}) \quad (3)$$

相应的边界条件为:

$$u = 0, v = v_w; r = a(t) \quad (4)$$

$$u = 0, v = 0; r = 0 \quad (5)$$

$$u = 0, v = 0; z = 0 \quad (6)$$

引入流函数 ψ :

$$\psi = \nu z F(\xi, t); \xi = \frac{r}{a} \quad (7)$$

把式(7)代入式(1)~式(3),可以得到一个四阶的偏微分方程:

$$\left(\frac{F_\xi}{\xi}\right)_{\xi\xi\xi} + \left(\frac{F_\xi}{\xi}\right)_{\xi\xi} \left(\frac{F}{\xi} + \frac{1}{\xi} + \alpha\xi\right) - \left(\frac{F_\xi}{\xi} + \frac{F}{\xi^2} + \frac{1}{\xi^2} - 3\alpha\right) \left(\frac{F_\xi}{\xi}\right)_\xi - \frac{a^2}{\nu} \left(\frac{F_\xi}{\xi}\right)_{\xi t} = 0 \quad (8)$$

S. Uchida 和 H. Aoki^[6] 对方程应用时间和空间上的相似变换,并假设壁面的膨胀率 $\alpha = aa'/v$ 为常数,则式(8)可变为:

$$\left(\frac{F_\xi}{\xi}\right)_{\xi\xi\xi} + \left(\frac{F_\xi}{\xi}\right)_{\xi\xi} \left(\frac{F}{\xi} + \frac{1}{\xi} + \alpha\xi\right) - \left(\frac{F_\xi}{\xi} + \frac{F}{\xi^2} + \frac{1}{\xi^2} - 3\alpha\right) \left(\frac{F_\xi}{\xi}\right)_\xi = 0 \quad (9)$$

边界条件式(4)~式(6)应为:

$$F(1) = Re, F'(1) = 0,$$

$$F(0) = 0, \lim_{\xi \rightarrow 0} \left[\frac{F_{\xi\xi}}{\xi} - \frac{F_\xi}{\xi^2} \right] = 0 \quad (10)$$

引入变换: $\eta = \xi^2$, 此时式(9)变为

$$8\eta F'' + 4FF'' + 8F' + \alpha(4\eta F'' + 4F') - 4(F')^2 = K \quad (11)$$

其中 K 是积分常数,一撇代表对 η 求导。令 $f = F/Re$, 式(11)变为

$$\eta f'' + f' + \frac{\alpha}{2}(\eta f' + f') + \frac{Re}{2}(ff'' - (f')^2) = k \quad (12)$$

这里 $k = K/(8Re)$, 式(10)变为

$$f(1) = 1, f'(1) = 0,$$

$$f(0) = 0, \lim_{\eta \rightarrow 0} \eta^{1/2} f''(\eta) = 0 \quad (13)$$

式(12)~式(13)形成了一个含有多参数奇异点 $\eta=0$ 的边值问题。

1 对应大喷注 ($Re \rightarrow +\infty$) 的渐近解

当圆形管道壁面是静止时, R. M. Terrill^[7-9] 解决了此类问题的奇异边值问题, 当圆形管道壁面是运动时构造渐近解。

令 $\varepsilon = \frac{2}{Re}$, 则式(12)可表示为

$$\varepsilon(\eta f''' + f'') + \varepsilon \frac{\alpha}{2}(\eta f'' + f') + ff'' - f'^2 = k\varepsilon = \lambda \quad (14)$$

在这里, 将构造 type I 和 type II 的解。为了实现这个目标, 将分析这些解的特性目的就是构造出渐近解。由数值解可看出, I 的解是抛物型, 鉴于已有的文献构造摄动解的文献和经验来说, 仅仅只需要使用正则摄动技术就能将解构造出来。然而 type II 的渐近解不好构造, 因为在管道中央存在边界层, 从流体力学角度来看待这个问题, 那就是在同圆形管道中心附近粘性力同惯性力一样起重要作用。因此, 在这奇异摄动技术是用于构造 II 的解, 对于 type I 和 type II 的解构造过程分别在下面详细展现。

1.1 type I 的渐近解

采用 lighthill method^[10] 方法求边值问题的渐近解, 在这里只展现如下结果。

$$f(\eta) = g(\xi) = \sum_{i=0}^{\infty} g_i(\xi) \varepsilon^i$$

$$\eta = \xi + \varepsilon X_1(\xi) + o(\varepsilon)$$

$$X_1(\xi) = \frac{\xi}{2} \sin\left(\frac{\pi}{2}\xi\right) - \left(1 + \frac{\alpha}{2}\right) \frac{\xi}{\pi} \cos\left(\frac{\pi}{2}\xi\right) + \frac{2}{\pi^2} \sin\left(\frac{\pi}{2}\xi\right) + \frac{1}{\pi} \cos\left(\frac{\pi}{2}\xi\right) + \left(\frac{1}{\pi} - \frac{2}{\pi^2} - \frac{1}{2}\right) \xi$$

$$g_0(\xi) = \sin\left(\frac{\pi}{2}\xi\right), g_1(\xi) = \frac{1}{2} \cos\left(\frac{\pi}{2}\xi\right)$$

1.2 type II 的渐近解

在这一部分将使用正则摄动方法构造内解和外解的展开式。为了构造外解,用常数变易法,方便与内解匹配。

$$f(\eta) = \sum_{i=0}^{\infty} f_i(\xi) \varepsilon^i = f_0 + f_1 \varepsilon + o(\varepsilon) \quad (15)$$

$$\lambda = \sum_{i=0}^{\infty} \lambda_i \varepsilon^i = \lambda_0 + \lambda_1 \varepsilon + o(\varepsilon) \quad (16)$$

将式(15)和式(16)代入式(12),收集 ε^p ($p=0, 1, 2, \dots$) 的系数,得到如下方程

$$\varepsilon^0: f_0 f_0'' - (f_0')^2 = \lambda_0 \quad (17)$$

$$\varepsilon^1: f_0 f_1'' - 2f_0' f_1' + f_0 f_1'' + \eta f_0'' + f_0'' + \frac{\alpha}{2} (\eta f_0'' + f_0') = \lambda_1 \quad (18)$$

式(17)和式(18)满足边值条件

$$f_0(1) = 1, f_n(1) = 0, f_n'(1) = 0$$

由于边界层的原因,在内解中利用 $f(0) = 0$, 作为外解式(15)是有效的。为了简化,只给出 f_0, f_1 的表达式:

$$f_0 = \sin \frac{\pi}{2} \eta,$$

$$f_1(\lambda, \eta) = k_0(\theta) (\sin \theta - \theta \cos \theta) + k_1(\theta) \cos \theta$$

其中 $\theta = \frac{\pi}{2} \eta$, 常数 λ 在内解匹配时将定义引入一种合适的变换来获得内解

$$\eta = \varepsilon^r X, f(\eta) = \varepsilon^p g(X)$$

代入式(14)中得

$$\varepsilon^{1+p} (Xg''' + g'') + \varepsilon^{1+p+r} \frac{\alpha}{2} (Xg'' + g') + \varepsilon^{2p} (gg'' - (g')^2) = \lambda \varepsilon^{2r} \quad (19)$$

考虑粘性项和小膨胀比,令 $p=1, (\varepsilon^{1+p} = \varepsilon^{2p}), r=1$, 那么式(19)变为

$$Xg''' + g'' + \frac{\alpha \varepsilon}{2} (Xg'' + g') + gg'' - g'^2 = \lambda \quad (20)$$

式(20)的数值解不容易获得。因此

假设式(20)的摄动解的形式为:

$$g(x) = \sum_{i=0}^{\infty} g_i(x) \varepsilon^i,$$

则有如下方程

$$xg_0''' + g_0'' + g_0 g_0'' - g_0'^2 = \lambda_0 \quad (21)$$

$$xg_1''' + g_1'' + \frac{\alpha}{2} (xg_0'' + g_0') + g_0 g_1'' -$$

$$2g_0' g_1' + g_0 g_1' = \lambda \quad (22)$$

式(21)和式(22)的解形如幂级数解 $g_0(x) =$

$\sum_{i=0}^{\infty} a_n x^n$ 和 $g_1(x) = \sum_{i=0}^{\infty} b_n x^n$, 在管道中心运用边值条件,得到

$$g_0(x) = a_1 x + [a_1^2 - (\frac{\pi}{2})^2] \frac{x^2}{2} +$$

$$a_1 [a_1^2 - (\frac{\pi}{2})^2] \frac{x^3}{12} + [a_1^2 - (\frac{\pi}{2})^2]^2 \frac{x^4}{72} + \dots$$

$$g_1(x) = \frac{\alpha a_1 - 2\lambda_1}{4a_1} x + \frac{1}{2a_1} (a_1^2 - \frac{\pi^2}{4}) (\frac{3}{4} \alpha + \frac{\lambda_1}{2a_1}) x^2$$

由此可得:

$$g_0(x) = \frac{\pi}{2} x - 3 + 3e^{-\frac{\pi}{2}x} \quad (23)$$

$$g_1'' + \pi g_1' + \frac{\pi^2}{4} g_1 + \frac{\alpha}{2} (\frac{\pi^2}{4} x - \frac{\pi}{2}) = 0 \quad (24)$$

$$(\frac{\pi}{2} x - 3) g_1'' - \pi g_1' + \frac{\alpha \pi}{4} + x g_1''' + g_1'' = \lambda_1 \quad (25)$$

由式(24)和式(25)得:

$$g_1'' + \frac{\pi}{2} g_1' = \frac{2\alpha \pi - \alpha \pi^2 - 8\lambda_1}{4\pi x + 16}$$

由 $g_1(0) = 0$ 得

$$g_1(x) = -\frac{\alpha}{2} x + (\frac{3\alpha}{\pi} - \frac{4\lambda_1}{\pi^2}) \ln(\frac{\pi}{4} x + 1) +$$

$$C_1 + \frac{\alpha}{\pi} - \frac{3\alpha \pi - 4\lambda_1}{\pi} \int \frac{e^{\frac{\pi}{2}x}}{x\pi + 4} dx \cdot e^{-\frac{\pi}{2}x} +$$

$$(-C_1 - \frac{\alpha}{\pi} + \frac{3\alpha \pi - 4\lambda_1}{\pi} \int \frac{e^{\frac{\pi}{2}x}}{\pi x + 4} dx |_{x=0}) e^{-\frac{\pi}{2}x}$$

(26)

外解需要求解如下方程组

$$\begin{cases} (\sin \theta - \theta \cos \theta) k_0'(\theta) + \cos \theta k_1'(\theta) = 0 \\ \theta \sin^2 \theta k_0'(\theta) - k_1'(\theta) \sin^2 \theta = \frac{4\lambda_1}{\pi^2} + \\ \theta \cos \theta + \sin \theta - \frac{\alpha}{\pi} (\cos \theta - \theta \sin \theta) \end{cases}$$

解得

$$k_0(\theta) = -\frac{2\lambda_1}{\pi^2} \cot^2 \theta - \frac{1}{2} \theta \csc \theta \cot \theta -$$

$$\frac{3}{2} \csc \theta - \frac{1}{2} \int \theta \csc \theta d\theta + \frac{\alpha}{\pi} \left(\frac{1}{2} \csc \theta \cot \theta - \frac{1}{2} \int \csc \theta d\theta + \theta \csc \theta \right) + C_1 \quad (27)$$

$$k_1(\theta) = \frac{2\lambda_1}{\pi^2} (\theta \csc^2 \theta - \cot \theta) - \frac{\theta \cos \theta \cot \theta}{2} - 2\theta \csc \theta + 2 \int \csc \theta d\theta - \frac{1}{2} \int \theta^2 \csc \theta d\theta + \frac{\alpha}{\pi} \left(\frac{1}{2} \theta \csc \theta \cot \theta + \frac{1}{2} \int \theta \csc \theta d\theta + \int \csc \theta d\theta - \theta \csc \theta + \frac{1}{2} \csc \theta \right) + C_2 \quad (28)$$

其中

$$\int \csc \theta d\theta = \frac{1}{2} \ln \left| \frac{1 - \cos \theta}{1 + \cos \theta} \right|$$

$$\int \theta \csc \theta d\theta = \theta + \sum_{k=1}^{\infty} \frac{2(2^{2k-1} - 1) |B_{2k}|}{(2k+1)!} \theta^{2k+1}$$

$$\int \theta^2 \csc \theta d\theta = \frac{1}{2} \theta^2 + \sum_{k=1}^{\infty} \frac{2(2^{2k-1} - 1) |B_{2k}|}{(2k+1)!} \theta^{2k+2}$$

$$\text{令 } \theta = \frac{\pi}{2} \eta, f'(\eta) = \frac{\pi^2}{4} h''(\theta)$$

$h(\theta) = k_0(\theta)(\sin \theta - \theta \cos \theta) + k_1(\theta) \cos \theta$
则有:

$$h'(\theta) = k_0'(\theta) \sin \theta - k_1(\theta) \sin \theta$$

由 $h'(\frac{\pi}{2}) = 0$ 得 $k_1(\frac{\pi}{2})$

$$h''(\theta) = k_0''(\theta) \sin \theta + k_0'(\theta) (\sin \theta + \theta \cos \theta) - k_1'(\theta) \sin \theta - k_1(\theta) \cos \theta \quad (29)$$

把式(27)和式(28)代入式(29)中得

$$h''(\theta) = \frac{2\lambda_1}{\pi^2} (\theta \cos \theta - 2\theta \csc \theta \cot \theta + 2 \csc \theta) - \theta^2 \cot^2 \theta - \frac{3}{2} + \frac{\alpha}{\pi} \left[\left(-\frac{1}{2} \theta \cos \theta - \frac{1}{2} \sin \theta - \cos \theta \right) \int \csc \theta d\theta - \frac{\cos \theta}{2} \int \theta \csc \theta d\theta + \theta^2 \cot \theta + \theta \cot \theta - \cot \theta + 2\theta \right] + \frac{\cos \theta}{2} \int \theta^2 \csc \theta d\theta - \frac{\sin \theta + \theta \cos \theta}{2} \int \theta \csc \theta d\theta - 2 \cos \theta \int \csc \theta d\theta + C_1 (\theta \cos \theta + \sin \theta) + C_2 \cos \theta$$

$$h''(\frac{\pi}{2}) = \frac{4\lambda_1}{\pi^2} - \frac{3}{2} + \alpha - \frac{1}{2} \int \theta \csc \theta d\theta \Big|_{\theta=\frac{\pi}{2}} + C_1$$

$$f''_1(1) = \frac{\pi^2}{4} h''(\frac{\pi}{2}) = -\frac{\pi^2}{4} + \varepsilon(\lambda_1 - \frac{\pi^2}{8} \alpha)$$

$$f = f_0 + \varepsilon f_1 + \varepsilon^2 f_2$$

$$f'(1) = f'_0(1) + \varepsilon f'_1(1) + \dots$$

$$= -\frac{\pi^2}{4} + \varepsilon(\lambda_1 - \frac{\pi^2}{8} \alpha)$$

对于内解: $\eta = \varepsilon x, f(\eta) = \varepsilon g(x)$,

$$g(x) = g_0(x) + \varepsilon g_1(x) + \varepsilon^2 g_2(x) + \dots$$

$$g_0(x) = \frac{\pi}{2} x - 3 + 3e^{-\frac{\pi}{2}x}$$

$$f(\eta) = \varepsilon g(x) = \frac{\pi \eta}{2} + \left[\left(\frac{1}{4} \alpha - \frac{\lambda_1}{\pi} \right) \eta - 3 \right] \varepsilon + \left(C_1 + \frac{\alpha}{\pi} \right) \varepsilon^2$$

对于外解:

$$f(\eta) = f_0(\eta) + \varepsilon f_1(\eta) + \varepsilon^2 f_2(\eta) + \dots$$

其中 $f_0 = \sin \frac{\pi}{2} \eta$

$$f_1(\eta) = k_0(\theta) (\sin \theta - \theta \cos \theta) + k_1(\theta) \cos \theta$$

$$f'(1) = \frac{\pi^2}{4} h''(\frac{\pi}{2}) = -\frac{\pi^2}{4} + \varepsilon(\lambda_1 - \frac{\pi^2}{8} \alpha)$$

其中

$$\varepsilon = \frac{2}{Re}, \lambda_1 = -9.61, f''(1) = -2.4674 - \frac{19.22}{Re} \quad (30)$$

$$\text{当 } \alpha = 2, f'(1) = -2.4674 - \frac{24.578}{Re} \quad (31)$$

$$\text{当 } \alpha = -2, f'(1) = -2.4674 - \frac{14.2852}{Re} \quad (32)$$

由式(30)~式(32)可得 $-f''(1)$ 的数值解与渐近解比较,详见表1。

表1 $-f''(1)$ 的数值解与渐近解比较

Table 1 Comparison of numerical and asymptotic solutions

Re	$\alpha=0$		$\alpha=-2$		$\alpha=2$	
	数值解	渐近解	数值解	渐近解	数值解	渐近解
60	2.788 0	2.787 7	2.711 1	2.705 5	2.863 4	2.870 0
70	2.741 2	2.742 0	2.611 2	2.671 5	2.812 3	2.812 5
80	2.707 5	2.707 7	2.645 9	2.646 0	2.769 1	2.769 3
90	2.681 1	2.681 0	2.626 0	2.626 1	2.735 4	2.735 8
100	2.659 6	2.659 6	2.610 2	2.610 3	2.708 8	2.708 9

$$\lambda = (xg''' + g'') + \varepsilon \frac{\alpha}{2} (xg'' + g') + gg'' - (g')^2 \quad (33)$$

$$g = g_0 + \varepsilon g_1 + \varepsilon^2 g_2 + \dots \quad (34)$$

由式(33)和式(34),收集 ε^2 的系数,得:

$$(xg_2''' + g_2'') + \varepsilon \frac{\alpha}{2}(xg_1'' + g_1') + g_0 g_2'' + g_1 g_1' + g_0 g_2 - 2g_0' g_2' - (g_1')^2 = \lambda_2 \quad (35)$$

由式(23)、式(26)、式(34)、式(35)得

$$g_0''(0) = \frac{3\pi^2}{4} + \varepsilon \left(-\frac{\pi^2}{2} - \frac{\alpha\pi}{2} + \lambda_1 \right) + \varepsilon^2 \left(\frac{\alpha\pi C_1}{12} - \lambda_2 \right)$$

当 $\lambda_1 = 9.61, Re = 24.815, f''(0) = 132.2$ 时

$$C_1 = -9.3051807, \lambda_2 = -50.224824$$

$$f_0'(0) = 3.7011Re + \frac{100.449648}{Re} - \frac{\alpha\pi}{2} + 36.3092246 \quad (36)$$

由式(36)可得 $f''(0)$ 的数值解与渐近解比较,详见表2。

表2 $f''(0)$ 的数值解与渐近解比较

Table 2 Comparison of numerical and asymptotic solutions

Re	$\alpha=0$		$\alpha=-2$		$\alpha=2$	
	数值解	渐近解	数值解	渐近解	数值解	渐近解
40.367	188.112 4	188.200 0	191.452 0	191.583 0	184.832 1	184.817 0
55.817	244.687 7	244.693 2	248.009 2	248.009 4	241.374 5	241.377 1
66.034	282.223 2	282.229 0	285.512 1	285.518 1	278.938 9	278.939 8
73.034	309.949 8	309.949 9	313.224 3	313.224 0	306.675 8	306.675 9

2 结 论

本文基于牛顿流体在一个半无限长胀缩渗透圆形管道内流动的物理模型。当大喷注时一些相应的渐近解被构造出来,且渐近解与数值解的结果吻合的很好。

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