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次扩散机制下带有交易成本的 Merton 期权定价模型

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摘　要:研究了次扩散过程驱动下带有交易成本的 Merton 期权定价模型.得到了此模型下欧式看涨期权所满足的 Black-Scholes 方程,并给出了欧式看涨期权的定价公式.

关键词:次扩散过程;交易成本;期权定价

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Pricing Options Under Merton Model with Transaction Costs in Subdiffusive Regime

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Abstract:In this paper,the Merton model of option pricing with transaction costs in subdiffusive regime is studied.The Black-Scholes equation which European call options satisfy is derived.Moreover,the explicit formula for european call option is given.

key words:subdiffusive process;transaction costs;option pricing

经典的 Black-Scholes[1](布莱克—斯克尔斯)期权定价模型包含两类资产：风险资产(例如,一支股票)和有固定收益率的无风险债券. 在此模型中，股票的价格服从如下的几何布朗运动

$$\mathrm{d}S_t = \mu S_t \mathrm{d}t + \sigma S_t \mathrm{d}B(t),$$

这里 $B(t)$ 是标准布朗运动,期望回报率 μ 和波动率 σ 都是常数.

虽然经典的 Black-Scholes 模型仍旧广泛应用于金融衍生产品的的定价中，但是在过去的几十年中，许多实证研究表明股票价格波动的许多典型特点是它所不能描述的，例如，重尾倾斜的边缘分布(heavy tailed and shewed marginal distributions),长期相关性(long range correlations)，常值周期性(periods of constant values)等. 因此许多学者选择用其他模型来代替经典的 Black-Scholes 模型[2-5].

在没有交易或交易量非常小的新兴市场以及利率、货币和商品市场中,资产价格变化经常出现常值周期性的特征[6-7].为了刻画金融数据中出现

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的这一现象,M.Magdziarz[8]把次扩散过程应用于期权定价问题研究中,提出了次扩散 Black-Scholes 模型.该模型中假设标的资产的价格:

$$S_\alpha(t)=S(T_\alpha(t))=S_0\mathrm{e}^{\mu T_\alpha(t)+\sigma B(T_\alpha(t))},$$

这里 $\alpha\in(0,1)$,$T_\alpha(t)$ 为逆 α-稳定从属过程,$B(\tau)$ 为标准布朗运动.他证明了在此模型下金融市场是无套利但不完备的,并给出了在此模型下的欧式期权定价公式.

此外,大量实证研究表明[9-10],在股票和外汇市场中,标的资产价格的运动轨迹是不连续的而且存在跳跃性.Merton(莫顿)[10]首先研究了跳扩散模型下的期权定价问题.本文研究当标的股票的价格服从下面的随机微分方程

$$S_t=S_0\mathrm{e}^{\mu T_\alpha(t)+\sigma B(T_\alpha(t))+N_t\ln J} \tag{1}$$

时且带有交易成本的欧式期权定价问题.这里 S_0,μ 和 σ 都是常数,$B(\tau)$ 是标准的布朗运动.$\alpha\in\left(\frac{1}{2},1\right)$,$T_\alpha(t)$ 为逆 α-稳定从属过程.N_t 是跳跃强度为 $\lambda>0$ 的泊松过程,J 非负的随机变量.这里假设 $T_\alpha(t)$,$B(\tau)$ 和 J 都是相互独立的.

1　带有交易成本的欧式期权定价

在下列假设条件下得到当股票价格 S_t 满足式(1)时,欧式看涨期权的定价公式.

(i) 标的股票的价格满足式(1),且 $\alpha\in\left(\frac{2}{3},1\right)$,$J$ 是取正值的随机变量且 J 的对数服从标准差为 σ' 的正态分布.

(ii) 存在交易费用,且交易费用与交易额成正比例. 即在股票价格为 S_t 时买入或卖出 D 份($D>0$, 买入;$D<0$, 卖出)时, 需要支付的交易费用为 $\frac{k}{2}|D|S_t$ 其中 k 为常数, 它的值取决于单个的投资者. 此外, 交易指发生在 t 和 $t+\Delta t$ 时刻, 在区间(t, $t+\Delta t$)里不发生交易. 这就意味着股票当前的价格 S_t 及份额在区间$[t,t+\Delta t)$里保持不变.

(iii) 投资组合 Π 由 $D(t)=D(t,S_t)$ 份股票和价值为 $Q(t)$ 的无风险债券构成, $\Pi_t=D(t)S_t+Q(t)$.Π_t 每隔 Δt 时间调整一次, 其中 Δt 是固定的小时间段.

(iv) 与离散时间无交易成本时的情况相同,这里假设投资组合的预期收益率与期权的预期收益率相同.

假设无风险债券的收益率为常数, 即

$$\Delta Q_t=rQ_t\Delta t+o(\Delta t),$$

这里无风险利率 r 为常数.

引理 1[11]　$\Delta T_\alpha(t)=o(\Delta t^{\alpha-\varepsilon})$,$\Delta B(T_\alpha(t))=o(\Delta t^{\frac{1}{2}\alpha-\varepsilon})$.

现在,令 $V=V(t,S_t)$ 表示在 t 时刻,标的股票价格为 S_t 时欧式看涨期权的价格,在假设条件(i)~(iv)下有:

定理 1　期权价格 $V=V(t,S_t)$ 满足下面的偏微分方程:

$$\frac{\partial V}{\partial t}=rS\frac{\partial V}{\partial S}+\frac{\tilde{\sigma}^2}{2}S^2\frac{\partial^2 V}{\partial S^2}+\lambda E[V(t,JS)-V(t,S)]-\lambda E[J-1]S\frac{\partial V}{\partial S}=0,$$

边界条件为

$$V(T,S_T)=\max\{S_T-K,0\}.$$

进一步,欧式看涨期权的价格为

$$V(t,S_t)=\sum_{n=0}^{\infty}\frac{1}{n!}\mathrm{e}^{-\lambda'(T-t)}(\lambda'(T-t))^n\times V_{BS}(t,S_t,\sigma_n,r_n),$$

其中,

$$\tilde{\sigma}=\left[\sigma^2\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)+\frac{kS_t^2}{\sqrt{2\pi}}\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)^{\frac{1}{2}}(\Delta t^{\frac{\alpha}{2}-1})\right],$$

$$\sigma_n^2=\tilde{\sigma}^2+\frac{n\sigma'^2}{T-t},r_n=r-\lambda E(J-1)+\frac{n\ln(EJ)}{T-t},\lambda'=\lambda EJ.$$

证明

1) $\Delta N_t=0$ 的概率为 $1-\lambda\Delta t$.由时间步长 Δt 非常小,则由泰勒公式可知,S_t 在长度为 Δt 的时间区间$[t,t+\Delta t)$上的增量为:

$$\begin{aligned}\Delta S_t&=S_{t+\Delta t}-S_t=S_t(\mathrm{e}^{\mu\Delta T_\alpha(t)+\sigma\Delta B(T_\alpha(t))}-1)\\&=S_t\Big[\mu\Delta T_\alpha(t)+\sigma\Delta B(T_\alpha(t))+\frac{1}{2}(\mu\Delta T_\alpha(t)+\sigma\Delta B(T_\alpha(t)))^2\Big]+\\&\quad\frac{1}{6}\mathrm{e}^{\theta(\mu\Delta T_\alpha(t)+\sigma\Delta B(T_\alpha(t)))}S_t(\mu\Delta T_\alpha(t)+\sigma\Delta B(T_\alpha(t)))^3,\end{aligned}$$

这里 $\theta=\theta(t,\omega)$ 是与过程 S_t 相关的一个随机变量,$\omega\in\Omega,\theta\in(0,1)$.

由假设 $\alpha>\frac{2}{3}$ 及引理 1:

$$\frac{1}{2}(\mu\Delta T_\alpha(t)+\sigma\Delta B(T_\alpha(t)))^2=\frac{1}{2}\sigma^2(\Delta B(T_\alpha(t)))^2+o(\Delta t) \tag{2}$$

$$\frac{1}{6}\mathrm{e}^{\theta(\mu\Delta T_\alpha(t)+\sigma\Delta B(T_\alpha(t)))}S_t(\mu\Delta T_\alpha(t)+$$

$$\sigma\Delta B(T_\alpha(t)))^3=o(\Delta t^{\frac{3}{2}\alpha-\varepsilon})=o(\Delta t). \quad (3)$$

因此,由等式(2)和等式(3)可得:

$$\Delta S_t=\mu S_t\Delta T_\alpha(t)+\sigma S_t\Delta B(T_\alpha(t))+\frac{1}{2}\sigma^2 S_t(\Delta B(T_\alpha(t)))^2+o(\Delta t),$$

$$(\Delta S_t)^2=\mu S_t\Delta T_\alpha(t)+\sigma S_t\Delta B(T_\alpha(t))+\frac{1}{2}\sigma^2 S_t(\Delta B(T_\alpha(t)))^2+o(\Delta t),$$

进一步,

$$\Delta V(t,S_t)=\frac{\partial V}{\partial t}\Delta t+\frac{\partial V}{\partial S_t}\Delta S_t+\frac{1}{2}\frac{\partial^2 V}{\partial S_t^2}(\Delta S_t)^2+o(\Delta t),$$

$$\Delta D(t,S_t)=\frac{\partial D}{\partial t}\Delta t+\frac{\partial D}{\partial S_t}\Delta S_t+\frac{1}{2}\frac{\partial^2 D}{\partial S_t^2}(\Delta S_t)^2+o(\Delta t).$$

2) $\Delta N_t=1$ 的概率为 $\lambda\Delta t$, 假设在时间区间[$t+\Delta t$)内股票价格只在当前时刻 t 发生一次跳跃,有

$$S_{t+}=S_0\mathrm{e}^{\mu\Delta T_\alpha(t)+\sigma\Delta B(T_\alpha(t))+\ln J},$$

$$S_{t+\Delta t}=S_0\mathrm{e}^{\mu\Delta T_\alpha(t+\Delta t)+\sigma\Delta B(T_\alpha(t+\Delta t))+\ln J},$$

并且,

$$\Delta S_{t+}=S_{t+\Delta t}-S_{t+}=S_{t+}[\mathrm{e}^{\mu\Delta T_\alpha(t)+\sigma\Delta B(T_\alpha(t))}-1],$$

$$\Delta S_t=S_{t+\Delta t}-S_t=S_{t+\Delta t}-S_{t+}+S_{t+}-S_t=S_{t+}[\mathrm{e}^{\mu\Delta T_\alpha(t)+\sigma\Delta B(T_\alpha(t))}-1]+S_{t+}-S_t,$$

$$\Delta V(t,S_t)=V(t+\Delta t,S_{t+\Delta t})-V(t,S_{t+})+V(t,S_{t+}-V(t,S_t)=V(t,S_{t+}-V(t,S_t)+\frac{\partial V}{\partial t}\Delta t+\frac{\partial V}{\partial S_{t+}}\Delta S_{t+}+\frac{1}{2}\frac{\partial^2 V}{\partial S_{t+}^2}(\Delta S_{t+})^2+o(\Delta t),$$

$$\Delta D(t,S_t)=D(t,S_{t+})-D(t,S_t)+\frac{\partial D}{\partial t}\Delta t+\frac{\partial D}{\partial S_{t+}}\Delta S_{t+}+\frac{1}{2}\frac{\partial^2 D}{\partial S_{t+}^2}(\Delta S_{t+})^2+o(\Delta t),$$

由假设(iii)有

$$\Delta\Pi_t=D(t)\Delta S_t+\Delta Q_1-\frac{k}{2}|\Delta D(t)|S_t,$$

而由假设(iv)可知,

$$E(\Delta\Pi)=E(\Delta V), \quad (4)$$

又由泊松过程 N_t 的性质和关于跳跃的假设可得:

$$E(\Delta\Pi)=E(D(t)\Delta S_t+\Delta Q_t-\frac{k}{2}|\Delta D(t)|S_t)=(1-\lambda\Delta t)E[D(t)\Delta S_t+\Delta Q_t-\frac{k}{2}|\Delta D(t)|S_t]+\lambda\Delta tE(D(t)S_{t+}(\mathrm{e}^{\mu\Delta T_\alpha(t)+\sigma\Delta B(T_\alpha(t))}-1)+\Delta Q_t-\frac{k}{2}S_t|D(t,S_{t+})-D(t,S_t)+\frac{\partial D}{\partial t}\Delta t+\frac{\partial D}{\partial S_{t+}}\Delta S_{t+}+\frac{1}{2}\frac{\partial^2 D}{\partial S_{t+}^2}(\Delta S_{t+})^2+o(\Delta t)|], \quad (5)$$

$$\begin{aligned}E(\Delta V)&=(1-\lambda\Delta t)E[\Delta V(t,S_t)]+\lambda\Delta tE[\Delta V(t,S_t)]\\&=(1-\lambda\Delta t)E[\frac{\partial V}{\partial t}\Delta t+\frac{\partial V}{\partial S_t}\Delta S_t+\frac{1}{2}\frac{\partial^2 V}{\partial S_t^2}(\Delta S_t)^2+o(\Delta t)]+\\&\lambda\Delta tE[V(t,S_{t+})-V(t,S_t)+\frac{\partial V}{\partial t}\Delta t+\frac{\partial V}{\partial S_{t+}}\Delta S_{t+}+\frac{1}{2}\frac{\partial^2 V}{\partial S_{t+}^2}(\Delta S_{t+})^2+o(\Delta t)]\end{aligned} \quad (6)$$

在这里股票的当前价格 S_t 是已知的.

下面计算 $E\left(\frac{kS_t}{2}|\Delta D(t)|\right)$.

$$\begin{aligned}E\left(\frac{kS_t}{2}|\Delta D(t)|\right)&=\frac{(1-\lambda\Delta t)kS_t}{2}E(|\Delta D(t)|)+\frac{(\lambda\Delta t)kS_t}{2}E(|\Delta D(t)|)\\&=\frac{(1-\lambda\Delta t)kS_t}{2}E\left(\frac{\partial D}{\partial t}\Delta t+\frac{\partial D}{\partial S_t}\Delta S_t+\frac{1}{2}\frac{\partial^2 D}{\partial S_t^2}(\Delta S_t)^2+o(\Delta t)\right)+\\&\frac{(\lambda\Delta t)kS_t}{2}E(D(t,S_{t+})-D(t,S_t)+\frac{\partial D}{\partial t}\Delta t+\frac{\partial D}{\partial S_{t+}}\Delta S_{t+}+\frac{1}{2}\frac{\partial^2 D}{\partial S_{t+}^2}(\Delta S_{t+})^2+o(\Delta t))\\&=\frac{kS_t^2\sigma}{\sqrt{2\pi}}\left|\frac{\partial^2 V}{\partial S^2}\right|\left(\frac{t^{\frac{1}{2}\alpha-\frac{1}{2}}}{\Gamma(\alpha)}\right)(\Delta t)^{\frac{\alpha}{2}}+o(\Delta t^\alpha)\\&\approx\frac{kS_t^2\sigma}{\sqrt{2\pi}}\left|\frac{\partial^2 V}{\partial S^2}\right|\left(\frac{t^{\frac{1}{2}\alpha-\frac{1}{2}}}{\Gamma(\alpha)}\right)(\Delta t)^{\frac{\alpha}{2}}\end{aligned} \quad (7)$$

这里 $D(t,S_t)=\frac{\partial V}{\partial S_t}$.

则由等式(4)~式(7)可得

$$\frac{\partial V}{\partial t}+rS\frac{\partial V}{\partial S}+\frac{\sigma^2}{2}\frac{t^{\alpha-1}}{\Gamma(\alpha)}S^2\frac{\partial^2 V}{\partial S^2}-rV+\frac{kS_t^2\sigma}{\sqrt{2\pi}}\left|\frac{\partial^2 V}{\partial S^2}\right|\left(\frac{t^{\frac{1}{2}\alpha-\frac{1}{2}}}{\Gamma(\alpha)}\right)(\Delta t)^{\frac{\alpha}{2}}+\lambda E[V(t,JS)-V(t,S)]-\lambda E[J-1]S\frac{\partial V}{\partial S}=0 \quad (8)$$

记 $\Gamma=\frac{\partial^2 V}{\partial S^2}$,众所周知,对于没有交易成本的

欧式看涨或看跌期权来说 Γ 始终是正值,因而这里假设对所有的 S_t,Γ 不改变符号.

令

$$\widetilde{\sigma}^2(t)=\left[\sigma^2\left(\frac{t^{\alpha-1}}{\Gamma(\alpha)}\right)+\frac{kS_t^2\sigma}{\sqrt{2\pi}}\left(\frac{t^{\frac{1}{2}\alpha-\frac{1}{2}}}{\Gamma(\alpha)}\right)(\Delta t^{\frac{\alpha}{2}-1}sign(\Gamma)\right].$$

进而,如果 $\Gamma>0$ 由文献[10]我们可得

$$V(t,S_t)=\sum_{n=0}^{\infty}\frac{1}{n!}\mathrm{e}^{-\lambda'(T-t)}(\lambda'(T-t))^n V_{BS}(t,S_t,\sigma_n,r_n),$$

这里 $\sigma_n^2=\widetilde{\sigma}^2+\dfrac{n\sigma'^2}{T-t}$,$r_n=r-\lambda E(J-1)+\dfrac{n\ln(EJ)}{T-1}$,$V_{\mathrm{BS}}(\cdot)$ 为带跳的 Black-Scholes 公式,随机变量 J 的对数服从标准差为 σ' 的正态分布.

证毕

注:若出现跳跃时,标的股票的价格跳到零.此时即表现为当跳跃发生时随机变量 $J\equiv0$ 的概率为 1.因而 $E(J-1)=-1$,此时式(8)变成

$$\frac{\partial V}{\partial t}+(r+\lambda)S\frac{\partial V}{\partial S}+\frac{\sigma^2}{2}\frac{t^{\alpha-1}}{\Gamma(\alpha)}S^2\frac{\partial^2 V}{\partial S^2}-(r+\lambda)V+\frac{kS_t^2\sigma}{\sqrt{2\pi}}\left|\frac{\partial^2 V}{\partial S^2}\right|\left(\frac{t^{\frac{1}{2}\alpha-\frac{1}{2}}}{\Gamma(\alpha)}\right)(\Delta t^{\frac{\alpha}{2}-1})=0$$

当 $\Gamma>0$ 时,可得

$$\frac{\partial V}{\partial t}+(r+\lambda)S\frac{\partial V}{\partial S}+\frac{\widetilde{\sigma}^2(t)}{2}S^2\frac{\partial^2 V}{\partial S^2}-(r+\lambda)V=0. \tag{9}$$

此时可得欧式看涨期权的价格 $V(t,S)$ 为

$$V(t,S_t)=S_tN(d_1)-Ke^{-r(T-t)}N(d_2),$$

这里

$$d_1=\frac{\ln\left(\frac{S_t}{K}\right)+\left(r+\frac{\hat{\sigma}^2}{2}\right)(T-t)}{\hat{\sigma}\sqrt{T-t}},$$

$$d_2=d_1-\hat{\sigma}\sqrt{T-t},\hat{\sigma}^2=\frac{\int_t^T\widetilde{\sigma}^2(\tau)\mathrm{d}\tau}{T-t}.$$

从式(9)可知,当出现一次跳跃则股票价格为零时,原来的无风险利率 r 将由 $r+\lambda$ 来代替,这与 Merton 的结果是一致的.不同的是波动率 σ 不再是常数,σ^2 由 $\widetilde{\sigma}^2(t)$ 代替.

2　结　　论

本文运用期权定价理论和偏微分方程的方法研究了次扩散机制下带有交易成本的 Merton 期权定价模型.得到了此模型下欧式看涨期权所满足的 Black-Scholes 方程.并给出了欧式看涨期权的定价公式.与 Merton 的结果不同的是,本文所得到的欧式看涨期权的定价公式中波动率 σ 不再是常数,而是 $\widetilde{\sigma}^2(t)$.

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